

Dimensional reduction and gauge group reduction in Bianchi-Type cosmology

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In this paper we examine in detail the implementation, with its associated difficulties, of the Killing conditions and gauge fixing into the variational principle formulation of Bianchi-Type cosmologies. We address problems raised in the literature concerning the Lagrangian and the Hamiltonian formulations: We prove their equivalence, make clear the role of the Homogeneity Preserving Diffeomorphisms in the phase space approach, and show that the number of physical degrees of freedom is the same in the Hamiltonian and Lagrangian formulations. Residual gauge transformations play an important role in our approach, and we suggest that Poincaré transformations for special relativistic systems can be understood as residual gauge transformations. In Appendices, we give the general computation of the equations of motion and the Lagrangian for any Bianchi-Type vacuum metric and for spatially homogeneous Maxwell fields in a nondynamical background (with zero currents). We also illustrate our counting of degrees of freedom in an Appendix.

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I. INTRODUCTION

A spatially homogeneous cosmological model is a manifold \mathcal{M} with a Lorentzian metric tensor \mathbf{g} invariant under a group of isometries whose three-dimensional, spacelike, invariant hypersurfaces foliate \mathcal{M} . In the models we will treat, this group is generated by three spacelike vector fields \mathbf{K}_a , which span a Lie algebra defined by their commutation relations:

$$[\mathbf{K}_a, \mathbf{K}_b] = C_{ab}^c \mathbf{K}_c . \quad (1.1)$$

The invariance of the metric is expressed by the vanishing of its Lie derivatives with respect to these vectors: $\mathcal{L}_{\mathbf{K}_a} \mathbf{g} = 0$. The structure constants are antisymmetric in their lower indices and obey the Jacobi relation:

$$C_{ab}^c = -C_{ba}^c ; \quad C_{be}^a C_{cd}^e + C_{ce}^a C_{db}^e + C_{de}^a C_{bc}^e = 0 . \quad (1.2)$$

The Bianchi classification of these algebras into nine Types (see [1]) is the source of the term Bianchi-Type cosmology. In three dimensions the Jacobi relation is equivalent to

$$C_{ea}^a C_{bc}^e = 0 . \quad (1.3)$$

The algebras with vanishing C_{ea}^a are called Class A and those with nonvanishing C_{ea}^a are called Class B [2].

In a suitable basis, \mathbf{g} can be represented by components $g_{\mu\nu}$ which depend only on a single variable, cosmic time t . The Einstein field equations are (coupled, non-linear) ordinary equations. (The choice of this basis is part of this paper.) The Einstein-Hilbert Lagrangian density for the metric may be computed in terms of these components and their t -derivatives. It is well known that in Class A models this reduced Lagrangian does correctly reproduce the field equations, but it does not necessarily do so in Class B models [3–7] because of a spatial divergence that is automatically zero only in Class A [8].

Bianchi-Type cosmology is still a subject of debate and — we think — some misunderstanding. It has been claimed, surprisingly, that the number of degrees of freedom in the Lagrangian and Hamiltonian formulations of these models do not agree [9]. This result would mean that the two formulations are not physically equivalent. Another issue, related to this one, poses the question as to what is to be considered a gauge transformation, and therefore a redundancy in the physical description in these models.

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In this paper, we will address three subjects: the Lagrangian and Hamiltonian formulations of Bianchi-Type models (dimensional reduction), the concept of gauge freedom for these models, and the fact that the number of degrees of freedom in the Lagrangian and Hamiltonian formulations do agree. We do not solve the first subject, which remains a problem for Class B models, although we clarify some points that are also relevant for Class A models. From our analysis we give an answer to the second and third problems. We show that the gauge freedom is dictated by the diffeomorphism invariance of the original theory and leads to the physical equivalence of the Lagrangian and Hamiltonian formulations.

We treat homogeneous models in general relativity as constrained dynamical systems. In particular we clarify the two aspects that merge when we move from the superspace of all metrics to the minisuperspace of homogeneous metrics. On the one hand there is the dimensional reduction from $3+1$ spacetime dimensions to one time dimension. This dimensional reduction comes from the symmetry associated with the Lie algebra of the Killing vectors. On the other hand there is the elimination of gauge degrees of freedom. These two types of reduction present different types of problems, which we analyze, but both are necessary in order to reduce the original gauge group of four-diffeomorphism invariance to the gauge group of time reparameterization invariance.

This reduction procedure will be undertaken in four steps. As an outcome of our analysis, a) we will clarify some points concerning the Lagrangian and Hamiltonian reduced formulations; b) we will be able to point out clearly what is the gauge group for these models; and c) we will show that the correct reduction procedures in both the Lagrangian and Hamiltonian formulations lead to a number of degrees of freedom which is always the same in either formulation. The first step, in Section II, consists of adapting time coordinates to the symmetry group. The second step, in Section III, is to adapt spatial coordinates to the group by adopting a gauge in which the shift vector depends only on time. The third step, in Section IV, is to adopt a gauge in which the shift vector vanishes. Finally, the fourth step, in Section V, is to eliminate residual three-diffeomorphisms. After each step we examine the status of what remains of the gauge group. In Section VI we examine the loss of constraints caused by the gauge fixing. We perform, in Section VII, the Hamiltonian analysis, which is presented in a summary form because it is parallel to the Lagrangian analysis. Section VIII is devoted to conclusions.

In Appendix A we present explicit equations for spatially homogeneous metrics in full detail, including general lapse function and shift vector, for any Bianchi Type, where coordinates have been chosen so that all variables depend only on time. The purpose is to provide concrete examples for our formalism. A simpler example, which also can serve to illustrate our methods, is provided in Appendix B: It is the case of a spatially homogeneous electromagnetic field in a spatially homogeneous back-

ground metric. Appendix C reviews the general literature on the relationship between setting gauge conditions and variational principles. It gives justification to some results used in Section IV and Section VII. Appendix D illustrates our counting of degrees of freedom in the Bianchi Type I case and shows how spacetime rigid symmetries may be considered as residual gauge transformations.

II. TIME

We start with the general setting for Bianchi models: Call \mathbb{T} the set, each element of which is a metric tensor \mathbf{g} and three vector fields \mathbf{K}_a whose orbits generate three-dimensional hypersurfaces which foliate the four-dimensional manifold \mathcal{M} ; the Lie derivatives of \mathbf{g} with respect to the \mathbf{K}_a vanish. The invariance of \mathbf{g} and even the definition of the \mathbf{K}_a need only be locally defined for much of what we do; if the vector fields are globally defined, we speak of global homogeneous cosmologies [9].

The diffeomorphisms on \mathcal{M} can be viewed in an active sense, mapping points onto other points. In that case, an element (metric plus three vector fields) of \mathbb{T} will in general be mapped to another element in \mathbb{T} . If there is a covering of \mathcal{M} by coordinate patches, each diffeomorphism may also be viewed in a passive sense as a collection of coordinate transformations. It is this passive sense which is closer to the physical principle which is a motivation for General Relativity, that physics should not depend on coordinates. It is, in fact, somewhat easier to adopt the language of a metric being determined by its components in a coordinate patch; a diffeomorphism is then a transformation which preserves the metric \mathbf{g} and the vectors \mathbf{K}_a but in general changes their components $g_{\mu\nu}, K_a^\mu$. Nevertheless, we shall adopt the active view when convenient. (Greek indices range over 0,1,2,3, with the coordinate x^0 being the time t . Latin indices will range over 1,2,3.)

Given a coordinate neighborhood \mathcal{N} , we consider all metric tensors, defined by their components $g_{\mu\nu}$ in \mathcal{N} , which are invariant under isometries defined by the \mathbf{K}_a and in which the \mathbf{K}_a are spacelike. The metric components satisfy the Killing equation:

$$0 = (\mathcal{L}_{\mathbf{K}_a} \mathbf{g})_{\mu\nu} = K_a^\sigma g_{\mu\nu,\sigma} + g_{\sigma\nu} K_a^\sigma{}_{,\mu} + g_{\mu\sigma} K_a^\sigma{}_{,\nu}, \quad (2.1)$$

where comma denotes partial differentiation. We can consider \mathbb{T} as being the collection of all such tensor components, each element of \mathbb{T} being the collection of metric and Killing vector components, the coordinate system being understood. Any of the elements of \mathbb{T} foliates \mathcal{M} by three-dimensional space-like homogeneous hypersurfaces, namely the integral surfaces of the Killing vector fields.

A change of coordinates — a diffeomorphism — will in general change the form of the metric components. In

that sense, the diffeomorphism group $\mathbf{Diff}(\mathcal{M})$ on \mathcal{M} is realized as a group \mathbf{D} acting on \mathbb{T} . The physics doesn't change, of course, under a coordinate transformation, and symbolically we can write

$$\text{Physics (of a Bianchi Type)} = \mathbb{T}/\mathbf{D} .$$

These conditions need only hold locally for much of what we will be doing. In that sense, we really are working with a Lie algebra of infinitesimal isometries rather than a Lie group, though we will continue to speak of the isometries as a symmetry group. When we need to require global homogeneity, we will clearly specify so.

The existence of these isometries of the metric can be used to simplify greatly the equations of motion, which we take to be the vacuum Einstein equations. At this point, the full gauge group \mathbf{D} of General Relativity, generated by $\mathbf{Diff}\mathcal{M}$, is still operating in \mathbb{T} . We will be looking for a reduced Lagrangian, with the Killing conditions built in, capable of describing these Bianchi models. We do so in four steps: adapting time, adapting space, fixing the gauge by requiring the shift vector (to be defined later) to vanish, and fixing the residual gauge freedom.

We now proceed to the first step, which will partially fix the gauge (that is fixing coordinates) by concentrating on the three-dimensional homogeneous hypersurfaces. Suppose that the homogeneous hypersurfaces happen to coincide with the hypersurfaces $t = \text{const}$ in the coordinates defining an element of \mathbb{T} , $(g_{\mu\nu}, K_a^\sigma)$; then we introduce an equivalence relation by saying that another element of \mathbb{T} is equivalent to this one if it is produced by an element of \mathbf{D} , that is by a diffeomorphism. The set of these equivalence classes is \mathbb{T}_t , and the full group \mathbf{D} has been reduced to a smaller group \mathbf{D}_t , namely those diffeomorphisms which preserve the condition that the homogeneous hypersurfaces are defined by $t = \text{const}$.

In this first step, we have made a choice of spacetime local coordinates such that the surfaces Σ_t of constant time coincide with the foliation defined by the Killing vectors. The time coordinate is a function whose curl vanishes under Lie differentiation by any of the Killing vectors:

$$\mathcal{L}_{\mathbf{K}_a} dt = 0 . \quad (2.2)$$

Our Killing vectors now take the general form (where $\partial_i \equiv \partial/\partial x^i$)

$$\mathbf{K}_a = K_a^i(t, x) \partial_i . \quad (2.3)$$

Notice the possible time dependence of \mathbf{K}_a . (We denote the four spacetime coordinates t, x^i by t, x when they are used as the arguments of a function.)

A. Gauge group after the first step

In these spacetime coordinates $\{t, x^i\}$, a general infinitesimal reparameterization (coordinate transformation) is generated by a vector field $\epsilon^\mu(x) \partial_\mu$:

$$\begin{pmatrix} t \\ x^i \end{pmatrix} \longrightarrow \begin{pmatrix} t + \delta t \\ x^i + \delta x^i \end{pmatrix} = \begin{pmatrix} t + \epsilon^0 \\ x^i + \epsilon^i \end{pmatrix} . \quad (2.4)$$

To preserve equation (2.2), that is, to generate a foliation-preserving diffeomorphism, requires

$$\epsilon_{,i}^0 = 0 . \quad (2.5)$$

B. Expressing the homogeneous metric

Recall the commutation relations of the Killing vectors equation (1.1). At this point it is customary to introduce a set of three independent, right-invariant (under the Lie algebra) vector fields $\mathbf{Y}_a = Y_a^i(t, x) \partial_i$, which satisfy

$$[\mathbf{K}_a, \mathbf{Y}_b] = 0 . \quad (2.6)$$

We take them to be tangent to the homogeneous hypersurfaces. They also define a Lie algebra and can be taken such that

$$[\mathbf{Y}_a, \mathbf{Y}_b] = -C_{ab}^c \mathbf{Y}_c . \quad (2.7)$$

These commutation relations are ensured if we take \mathbf{Y}_a to coincide with \mathbf{K}_a at an arbitrary point in Σ_t and define them at other points in Σ_t by equation (2.6) [1].

In each Σ_t we can define a basis of 1-forms, $\omega^a = \omega_i^a(t, x) dx^i$, dual to these left-invariant vectors: $\omega^a(\mathbf{Y}_b) = \delta_b^a$. The Lie algebra property becomes (\mathbf{d}_3 is differentiation with respect to the space variables)

$$\mathbf{d}_3 \omega^a = \frac{1}{2} C_{bc}^a \omega^b \wedge \omega^c . \quad (2.8)$$

The metric can be written, using the anholonomic basis $\{dt, \omega^a\}$, as

$$\mathbf{g} = -N^2 dt^2 + g_{ab} (N^a dt + \omega^a) (N^b dt + \omega^b) , \quad (2.9)$$

where N is the lapse function and N^b the shift vector variables in this basis. The Killing conditions on \mathbf{g} give (\cdot denotes $\partial/\partial t$)

$$\mathbf{K}_a(g_{bc}) = 0 \iff g_{ab} = g_{ab}(t) , \quad (2.10a)$$

$$\mathbf{K}_a(N) = 0 \iff N = N(t) , \quad (2.10b)$$

$$\begin{aligned} \mathcal{L}_{\mathbf{K}_a} (N^b dt + \omega^b) &= 0 \\ \iff \dot{\mathbf{K}}_a &= [N^b \mathbf{Y}_b, \mathbf{K}_a] = -\mathbf{K}_a(N^b) \mathbf{Y}_b . \end{aligned} \quad (2.10c)$$

The first two results are clear: no spatial dependence for N and g_{ab} . The third result links the time dependence of the Killing vectors to the spatial dependence of the shift variables N^a . It relates dimensional reduction and

gauge-fixing reduction. Since the shift variables are completely arbitrary because of the gauge freedom, namely diffeomorphisms satisfying $\epsilon_{,i}^0 = 0$ (see equation (2.5)), we may consider this third relation as giving the time evolution of the Killing vector components for a given set of the shift variables. This shows in particular that if for an “initial” time, say $t = 0$, the Killing conditions are satisfied, they will be satisfied in future times, the third relation providing us with the form which the Killing vectors take.

At this point we could proceed to reduce the Lagrangian (which we will describe below). But reduction within the Lagrangian and reduction in the equations of motion are procedures that in general do not commute, as we could immediately verify. In the next section a thorough study of this non-commutativity will be given.

The fact that the shift variables depend arbitrarily on the space coordinates shows that we must further reduce the gauge freedom in order to end up with the reduced gauge group of time-reparameterization invariance. To do so, the simplest way is to require the shift variables to be spatially constant. According to the third relation above, this is equivalent to requiring time independence of the Killing vectors. This is our second step in the gauge fixing procedure, in the next section, and is a case of adapting spatial coordinates to the Killing structure.

III. SPACE

We introduce the partial gauge-fixing conditions

$$N_{,i}^a = 0. \quad (3.1)$$

We know from the previous analysis, equation (2.10c), that the Killing vectors become time independent,

$$\mathbf{K}_a = K_a^i(x) \partial_i,$$

and we can also choose the right invariant vectors \mathbf{Y}_a and the 1-forms ω^a to be so. It was realized in [10] that the shift vector $\mathbf{N} \equiv N^a(t) \mathbf{Y}_a$ is a one-parameter family of inner automorphisms of the right-invariant Lie algebra (generators \mathbf{Y}_b). More details are given in Section V. Now the metric is written as

$$\mathbf{g} = -N^2(t) \mathbf{d}t^2 + g_{ab}(t) (N^a(t) \mathbf{d}t + \omega^a) (N^b(t) \mathbf{d}t + \omega^b), \quad (3.2)$$

with

$$d\omega = \frac{1}{2} C_{bc}^a \omega^b \wedge \omega^c,$$

and the Killing conditions are built in.

We proceed to study the non-commutativity of the operations of introducing the conditions $g_{ab,i} = N_{,i} = N_{,i}^a = 0$ into the equations of motion (that is, to look for a restricted set of solutions of the Einstein equations) or

directly into the Lagrangian. The calculation given in Appendix A shows explicit examples of the ideas we discuss in this Section. In the anholonomic basis $\{\mathbf{d}t, \omega^a\}$, the Einstein-Hilbert Lagrangian density, once a total divergence is discarded, can be written as (our notations follow [11])

$$\mathcal{L} = |\omega| \sqrt{g} N ({}^3R + K_{ab} K^{ab} - K^2) \equiv |\omega| \tilde{\mathcal{L}}, \quad (3.3)$$

where $|\omega| = \det(\omega_i^a)$, $g = \det(g_{ab})$, $N \neq 0$ is the lapse function, 3R is the three-metric curvature, K_{ab} the extrinsic curvature

$$K_{ab} = \frac{1}{2N} (\dot{g}_{ab} - N_{a|b} - N_{b|a}), \quad (3.4)$$

and $K = e^{ab} K_{ab}$ its trace ($|$ denotes the spatial covariant derivative and e^{ab} is the matrix inverse of g_{ab}). Notice that \mathcal{L} is second order in the three-space derivatives.

One could also add other fields in the Lagrangian and implement the Killing conditions on them. For instance, in the Einstein-Maxwell case, the one-form gauge field $\mathbf{A} = A_0 \mathbf{d}t + A_a \omega^a$ must satisfy $A_{0,i} = A_{a,i} = 0$ (see Appendix B). If we label as X the generic variable $g_{ab}, N, N^a, A_0, A_a, \dots$, then $\tilde{\mathcal{L}}$ is a function $\tilde{\mathcal{L}}(X, \dot{X}, \mathbf{Y}_a X, \mathbf{Y}_a \mathbf{Y}_b X)$, where the last argument in $\tilde{\mathcal{L}}$ has $a \geq b$. Notice that $\mathbf{Y}_a \mathbf{Y}_b X$ contains not only second order but also first order spatial derivatives of X .

Let us write the Euler-Lagrange equations using the variables just displayed. The variation $\delta \mathcal{L}$ is

$$\begin{aligned} \delta \mathcal{L} = & \frac{\partial \mathcal{L}}{\partial X} \delta X + \frac{\partial \mathcal{L}}{\partial \dot{X}} \delta \dot{X} + \frac{\partial \mathcal{L}}{\partial \mathbf{Y}_a X} \delta(\mathbf{Y}_a X) \\ & + \sum_{a \geq b} \frac{\partial \mathcal{L}}{\partial \mathbf{Y}_a \mathbf{Y}_b X} \delta(\mathbf{Y}_a \mathbf{Y}_b X). \end{aligned}$$

Integration by parts, dropping the boundary terms, and repeated use of the relation

$$\mathbf{Y}_a(|\omega|) + |\omega| \partial_j Y_a^j = |\omega| C_{ad}^d, \quad (3.5)$$

yields the following form for the Euler-Lagrange equations:

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta X} = & |\omega| \left(\frac{\partial \tilde{\mathcal{L}}}{\partial X} - \partial_t \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{X}} - (\mathbf{Y}_a + C_{ac}^c) \left(\frac{\partial \tilde{\mathcal{L}}}{\partial \mathbf{Y}_a X} \right) \right. \\ & \left. + \sum_{a \geq b} (\mathbf{Y}_b + C_{bd}^d) (\mathbf{Y}_a + C_{ac}^c) \left(\frac{\partial \tilde{\mathcal{L}}}{\partial \mathbf{Y}_a \mathbf{Y}_b X} \right) \right). \quad (3.6) \end{aligned}$$

Due to the particular structure of equation (3.3), the third and fourth order spacetime derivatives in equation (3.6) will cancel, but this fact does not alter our discussion.

If we define the reduced Lagrangian

$$\mathcal{L}_R(X, \dot{X}) \equiv \tilde{\mathcal{L}}(X, \dot{X}, \mathbf{Y}_a X = 0, \mathbf{Y}_a \mathbf{Y}_b X = 0),$$

we end up with

$$\begin{aligned}
& \left(\frac{\delta \mathcal{L}}{\delta X} \right)_{(\partial_i X = \partial_{ij} X = 0)} \\
&= |\omega| \left(\frac{\delta \mathcal{L}_R}{\delta X} - C_{ac}^c \left(\frac{\partial \tilde{\mathcal{L}}}{\partial \mathbf{Y}_a X} \right)_{(\partial_i X = \partial_{ij} X = 0)} \right. \\
&\quad \left. + \sum_{a \geq b} C_{ac}^c C_{bd}^d \left(\frac{\partial \tilde{\mathcal{L}}}{\partial \mathbf{Y}_a \mathbf{Y}_b X} \right)_{(\partial_i X = \partial_{ij} X = 0)} \right). \quad (3.7)
\end{aligned}$$

equation (3.7) displays the non-commutativity between the two procedures: Reduction of the Lagrangian through the Killing conditions or reduction of the equations of motion. As we have mentioned before, this problem was identified long ago. The distinction between Class A and Class B Bianchi models is crucial in this respect, for Class A is characterized by the vanishing of the trace C_{ab}^b [2].

Summing up, we see that the implementation of the Killing conditions within the Lagrangian produces no harm, that is, commutativity holds in equation (3.7), for Class A models. For Class B models the equations of motion derived from \mathcal{L}_R are wrong when there is a contribution different from zero coming from the last two pieces in equation (3.7). One may wonder whether this means that the situation is hopeless if we want to have a Lagrangian formulation for Class B models. Perhaps in general, but in some special cases there may be *ad hoc* solutions. In Appendix B we show an example of such a case within homogeneous Maxwell theory.

A. Gauge transformations after the second step

At this stage we have time-independent Killing vectors. Let $\mathbf{E} = \epsilon^\mu \partial_\mu = \epsilon^0 \partial_0 + \epsilon^a Y_a^i \partial_i$ be a generator of a diffeomorphism which preserves this requirement as well as equation (2.5):

$$[\mathbf{E}, \mathbf{K}_a] = [\dot{\mathbf{E}}, \mathbf{K}_a] = 0, \quad \epsilon^0 = \epsilon^0(t), \quad (3.8)$$

where $\dot{} = \partial/\partial t$. The general form for the spatial components of \mathbf{E} is given by

$$\epsilon^a(t, x) = \varphi^a(t) + \zeta^a(x), \quad (3.9)$$

with $\varphi^a(t)$ and $\zeta^i(x)$ arbitrary functions of time and spatial coordinates respectively.

Suppose \mathbf{E}_1 and \mathbf{E}_2 are two generators satisfying equation (3.8):

$$[\dot{\mathbf{E}}_1, \mathbf{K}_a] = [\dot{\mathbf{E}}_2, \mathbf{K}_a] = 0.$$

By the Jacobi identity it is clear that $[[\dot{\mathbf{E}}_1, \dot{\mathbf{E}}_2], \mathbf{K}_a] = 0$, but it is not necessarily the case that $[[\mathbf{E}_1, \mathbf{E}_2], \mathbf{K}_a] = 0$. A closure process which is more general than commutation applies for these generators. The infinitesimal diffeomorphism generated by \mathbf{E}_1 changes the Killing vector fields (in the active view of diffeomorphisms), but it also changes the invariant basis vector fields and the second generator \mathbf{E}_2 as well. These changes, particularly

changes in the invariant basis vectors \mathbf{Y}_a , mean that the form of equation (3.9) cannot be expected to be invariant during the process of commutation. We will not pursue this matter further, for in the next step, where the shift vector is set to zero, the form of the gauge group once again becomes straightforward.

Notice that for Class A models, the first term on the right in equation (3.9) is a Noether symmetry for \mathcal{L}_R . Our Lagrangian \mathcal{L} , equation (3.3), differs from the Einstein-Hilbert Lagrangian \mathcal{L}_{EH} by a divergence

$$\mathcal{L} = \mathcal{L}_{EH} + |\omega| \sqrt{g} m^a|_a.$$

General reparameterization invariance under a diffeomorphism generated by $\mathbf{v} = \epsilon^\mu(x) \partial_\mu = \varphi^a(t) \mathbf{Y}_a$ produces the functional variation of \mathcal{L}_{EH} ,

$$\delta_{\mathbf{v}} \mathcal{L}_{EH} = \partial_\mu (\epsilon^\mu \mathcal{L}_{EH}).$$

Therefore,

$$\delta_{\mathbf{v}} \mathcal{L} = \partial_\mu (\epsilon^\mu \mathcal{L}_{EH}) + \delta_{\mathbf{v}} (|\omega| \sqrt{g} (\mathbf{Y}_a m^a + C_{ab}^b m^a)). \quad (3.10)$$

Let us now reduce these expressions, for Class A models, introducing our second step gauge fixing, $\partial_i X = \partial_{ij} X = 0$, where $X = g_{ab}, N, N^a, A_0, A_a, \dots$. The left side of equation (3.10) becomes

$$|\omega| \delta_{\mathbf{v}} \mathcal{L}_R.$$

The first term on the right in equation (3.10) is

$$\begin{aligned}
& (\partial_\mu (\epsilon^\mu \mathcal{L}))_{(\partial_i X = \partial_{ij} X = 0)} \\
&= f^a(t) \partial_i \left(|\omega| Y_a^i \tilde{\mathcal{L}} \right)_{(\partial_i X = \partial_{ij} X = 0)} = 0,
\end{aligned}$$

where we have used equation (3.5) in the last step. The second term on the right in equation (3.10) becomes

$$\begin{aligned}
& (\delta_{\mathbf{v}} (|\omega| \sqrt{g} m^a|_a))_{(\partial_i X = \partial_{ij} X = 0)} \\
&= |\omega| \mathbf{Y}_a (\delta_{\mathbf{v}} (\sqrt{g} m^a))_{(\partial_i X = \partial_{ij} X = 0)} = 0.
\end{aligned}$$

We end up with $\delta_{\mathbf{v}} \mathcal{L}_R = 0$; this is a gauge Noether symmetry provided by the reduced theory.

The second term on the right in equation (3.9) is not retrievable as a gauge symmetry from the reduced Lagrangian formulation (except when it is a copy, with constants φ^a , of the first term). However, we must still provide a gauge fixing for it. In step four of our gauge fixing procedure we will deal with it. It is obvious from equation (3.9) that the process of reducing the gauge group is not yet finished; we now continue this task.

IV. ZERO SHIFT

The remaining diffeomorphism invariance allows us to perform a new partial gauge-fixing by introducing a set of constraints that helps to eliminate some of the arbitrariness that exists in our equations of motion. To simplify the analysis we consider the case in which the metric is the only field in \mathcal{L} . Should other gauge fields be present, the dynamics could have additional arbitrariness.

Notice that the Lagrangian is independent of \dot{N} and \dot{N}^a . This fact implies that the dynamical evolution vector field operator in configuration-velocity space has [12,13] a term of the type (see Appendix C for clarification of this point)

$$\int d^3x \left(\eta^0(t, x) \frac{\partial}{\partial \dot{N}} + \eta^a(t, x) \frac{\partial}{\partial \dot{N}^a} \right), \quad (4.1)$$

with η^0, η^a arbitrary functions. The important consequence of this arbitrariness is that to fix the dynamics we must fix the values of N and N^a . Here we are only interested in a partial gauge-fixing: We only fix the shift functions. The simplest way of doing so is by introducing the gauge-fixing constraints $N^a = 0$. It is always possible to pass from an initial configuration with $N^a \neq 0$ to a final configuration with $N^a = 0$ by using diffeomorphisms satisfying the restrictions of equation (3.9); the geometric picture of this transformation is to make the curves in \mathcal{M} tangent to the normal vector of the Σ_t coincident with the curves generated by ∂_t .

This is the Lagrangian version of the partial gauge fixing. Stability of the new constraints under time evolution yields the new constraints $\dot{N}^a = 0$; then requiring stability again will make the arbitrary functions $\eta^a = 0$.

To complete the picture, let us go back to the second gauge fixing step, equation (3.1). Consider equation (4.1) again. Stabilization of $N_{,i}^a = 0$ implies $\dot{N}_{,i}^a = 0$. The stabilization of $\dot{N}_{,i}^a = 0$ yields $\eta_{,j}^a = 0$. By the same token, $\eta_{,j}^0$ vanishes as a consequence of the relation, which is also a gauge fixing, $N_{,i} = 0$, obtained in the first gauge fixing step. Therefore, from the point of view of the reduction of the equations of motion, the arbitrariness in the dynamical evolution vector field operator is described by four functions $\eta^0(t), \eta^a(t)$. The gauge freedom associated with this arbitrariness is given by $\epsilon^0(t), \varphi^a(t)$ in equation (2.5) and equation (3.9). Notice that the second term in equation (3.9) is unretrievable from the reduced dynamics. This is another way to verify the limitations of the reduced formalism under the Killing conditions already pointed out in the last paragraphs of Section III.

A. Gauge group after the third step

Once the shift functions N^a have been set to zero, the gauge group has been greatly reduced, for the only remaining diffeomorphisms still available to us are those

that, besides keeping the three-foliation, preserve the conditions $N^a = 0$. This requirement results in

$$0 = \epsilon^a. \quad (4.2)$$

The remaining diffeomorphisms therefore are such that ϵ^0 only depends on t (see equation 2.5), whereas the ϵ^a depend only on the three-space coordinates x^i (see equation 4.2). A nice picture emerges: The remaining gauge group has been factorized into two commuting subgroups, the group of time reparameterizations and the group of three-space diffeomorphisms.

Properly speaking, only the first group is still a gauge group; it is directly associated with the freedom that is left in the Lagrangian evolution operator and that is displayed in the arbitrariness of η^0 in equation (4.1) (η^a are zero in our particular gauge-fixing).

The group of three-space diffeomorphisms is not a gauge group because it does not have room for what is characteristic of the gauge freedom: to connect the members of a family of field configurations, all of which are solutions of the equations of motion, that share the same set of initial-value conditions. This group must be understood as describing a redundancy in the space of initial-conditions for our theories. We call this group a residual gauge group. A parallel case in electromagnetism is the residual gauge symmetry that is left after the introduction of the Lorentz gauge $\partial_\mu A^\mu = 0$: The transformation $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$ is a residual gauge symmetry if $\square \Lambda = 0$. In any case, one must take into account that the gauge fixing procedure is only finished when we have completely removed these residual gauge transformations [13]. Further comments on this important point are made in the last section of this paper.

We finish this section by observing that the comments raised in the previous section still apply here: Only the diffeomorphisms of the form $\epsilon^i = B^a \mathbf{Y}_a^i$, with B^a constant are obtainable as gauge Noether symmetries from the reduced Lagrangian in Class A models. In the language introduced in the next section, these vector fields $B^a \mathbf{Y}_a^i$ define the inner automorphisms for the right invariant Lie algebra (generators \mathbf{Y}_a)

V. RESIDUAL THREE-DIFFEOMORPHISMS

Now we are ready to perform the fourth step of the gauge fixing. Notice that the three-metric, $g_{ab}(t)\omega^a\omega^b$ on the surfaces Σ_t is also Killing with respect to the vectors \mathbf{K}_a . We are going to fix, in principle, the residual three-diffeomorphism group by working from now on with this basis of one-forms $\{\omega^a\}$. In the active view, a general three-diffeomorphism will drive $\mathbf{K}_a \rightarrow \mathbf{K}'_a$, $\omega^a \rightarrow \omega'^a$. The transformed metric $g_{ab}(t)\omega'^a\omega'^b \equiv g'_{ab}(t, x)\omega^a\omega^b$ will be Killing with respect to the new set \mathbf{K}'_a and will in general no longer be Killing with respect to the original vectors \mathbf{K}_a . In other words, g'_{ab} will in general acquire a dependence on the spatial coordinates. However, there

is another possibility to explore: the case when the new set \mathbf{K}'_a happens to belong to the Lie algebra generated by the original Killing vectors \mathbf{K}_a . This is equivalent to saying that

$$\omega'^a = M^a_b \omega^b, \quad (5.1)$$

with M^a_b constant. These are the homogeneity-preserving diffeomorphisms (HPD), which were introduced by Ashtekar and Samuel [9].

Generators \mathbf{v} for the HPD are better expressed in the basis of the invariant vectors \mathbf{Y}_a , $\mathbf{v} = f^a(x)\mathbf{Y}_a$. The condition for an HPD is (\mathcal{L} is Lie derivative here)

$$\mathcal{L}_{\mathbf{v}}\omega^b = B^b_a \omega^a, \quad (5.2)$$

with B^b_a a constant matrix. This is equivalent to

$$\mathcal{L}_{\mathbf{v}}\mathbf{K}_a = -A^b_a \mathbf{K}_b, \quad (5.3)$$

with A^b_a a constant matrix. The left-invariant Lie algebra (generators \mathbf{K}_a) and the right-invariant Lie algebra (generators \mathbf{Y}_a) possess the same automorphisms. If the point x_0 in Σ is the point where \mathbf{Y}_a coincides with \mathbf{K}_a , then $B^b_a = A^b_a + f^c(x_0)C^b_{ca}$.

Equation (5.3) is

$$\begin{aligned} (\mathbf{K}_a f^c)\mathbf{Y}_c &= -[\mathbf{v}, \mathbf{K}_a] = A^b_a \mathbf{K}_b \\ \iff (\mathbf{K}_a f^c) &= (\mathbf{K}_b \cdot \omega^c) A^b_a. \end{aligned} \quad (5.4)$$

This last equation is equivalent to

$$\mathbf{d}f^c = (\mathbf{K}_b \cdot \omega^c) A^b_a \mathbf{K}^a, \quad (5.5)$$

where \mathbf{K}^a are the dual forms to \mathbf{K}_b . Notice that $(\mathbf{K}_b \cdot \omega^c)$ is the adjoint representation of the Lie group expressed in the local patch we are working in. Because $\mathbf{d}\mathbf{K}^c = -\frac{1}{2}C^c_{ab}\mathbf{K}^a \wedge \mathbf{K}^b$ and $\mathbf{d}(\mathbf{K}_d \cdot \omega^c) = C^c_{bd}(\mathbf{K}_e \cdot \omega^c)\mathbf{K}^b$, the local integrability conditions for equation (5.5),

$$\mathbf{d}((\mathbf{K}_b \cdot \omega^c) A^a_b \mathbf{K}^a) = 0, \quad (5.6)$$

read

$$C^a_{eb}A^e_c - C^a_{ec}A^e_b + C^e_{bc}A^a_e = 0, \quad (5.7)$$

which are the conditions for Lie algebra automorphisms. Trivial solutions are $A^a_b = 0$ and $A^a_b = A^c C^a_{cb}$, with A^c constant. The solutions for the first case are

$$\mathbf{v} = B^c \mathbf{Y}_c, \quad (5.8)$$

with B^c constant, and for the second,

$$\mathbf{v} = A^c \mathbf{K}_c. \quad (5.9)$$

The first (second) solutions describe the inner automorphisms for the right-invariant (left-invariant) Lie algebra. In finite form, the M^a_b in equation (5.1), ($\mathbf{M} = e^{\mathbf{B}}$, \mathbf{B} of equation 5.2), satisfy

$$(\mathbf{M}^{-1})^d_a (\mathbf{M}^{-1})^e_b C^f_{de} M^c_f = C^c_{ab}. \quad (5.10)$$

The integrability conditions equation (5.7) only guarantee the local existence of the HPD. But we need these HPD to be global diffeomorphisms if equation (5.10) is to hold everywhere. Both the Lie algebra structure and the topology of the surfaces of homogeneity play a role in the determination of the existing HPD. In particular, for globally homogeneous cosmologies with simply connected surfaces of homogeneity, every solution of equation (5.7) defines a generator of a HPD. The relevant role of spatial topology was first emphasized in [9] and treated in great detail in [16–18]. The degrees of freedom for Class A globally homogeneous Bianchi models are also discussed in this reference; we will return to the question of degrees of freedom later.

A. Gauge group after the fourth step

Once we have completely eliminated the residual three-diffeomorphism gauge invariance, the remaining gauge group is that of time reparameterizations. The lapse variable transforms, under the time reparameterization gauge group, as a scalar density:

$$\delta_{\epsilon^0(t)} N = \epsilon^0 \dot{N} + \dot{\epsilon}^0 N, \quad (5.11)$$

and the three-metric variables g_{ab} as a scalar:

$$\delta_{\epsilon^0(t)} g_{ab} = \epsilon^0 \dot{g}_{ab}. \quad (5.12)$$

VI. CONSEQUENCES OF $N^a = 0$

We have seen in Section III the non-commutativity between the operations of inserting the Killing conditions into the equations of motion or directly into the Lagrangian. Here we are going to consider another kind of non-commutativity, the one coming from plugging the gauge fixing $N^a = 0$ into the Lagrangian. To make things clear, we only consider here the reduction of \mathcal{L} in equation (3.3) to a reduced Lagrangian \mathcal{L}_{GF} , prior to the introduction of the Killing conditions in it (GF stands for gauge fixing).

The important question is whether this new Lagrangian \mathcal{L}_{GF} is going to reproduce the same equations of motion derived from \mathcal{L} under the gauge-fixing conditions $N^a = 0$. The answer, from a general perspective which is summarized in Appendix C, can be found in [14], and in general is in the negative.

Bringing the results of [14] to our case, we find (here we use $[\mathcal{L}]$ for the Euler-Lagrange variation):

$$[\mathcal{L}] = 0, \quad N^a = 0 \iff [\mathcal{L}_{\text{GF}}] = 0, \quad \mathcal{H}_a = 0, \quad (6.1)$$

where \mathcal{H}_a are the Hamiltonian momentum constraints in the anholonomic basis and expressed in coordinate-velocity space. The N^a have been set to zero in \mathcal{H}_a . If we also implement the Killing conditions in \mathcal{H}_a , we get

$$\begin{aligned}\mathcal{H}_a &= -|\omega| \frac{\sqrt{g}}{N} (\dot{g}_{bc|a} - \dot{g}_{ac|b}) g^{bc} \\ &= |\omega| \frac{\sqrt{g}}{N} (C_{ab}^c k_c^b + C_{bc}^b k_a^c),\end{aligned}\quad (6.2)$$

where $k_c^b = g^{bd} \dot{g}_{dc}$.

Therefore, if we use the completely reduced Lagrangian, with gauge fixing plus Killing conditions, with configuration variables N, g_{ab} , we must be aware that the correct equations of motion will require that initial conditions be taken such that

$$C_{ab}^c k_c^b + C_{bc}^b k_a^c = 0. \quad (6.3)$$

We show in Appendix C that if the initial conditions satisfy equation (6.3), then equation (6.3) will hold at any time, provided the equations of motion are satisfied.

In Class A models, where $C_{bc}^b = 0$, equation (6.3) has only the term $C_{ab}^c k_c^b = 0$. It is worth noting that, except for Type I models, which have vanishing structure constants, in all other Class A models, equation (6.3) must be enforced on the initial conditions. A diagonal form for the metric, for instance, will guarantee the fulfillment of equation (6.3) for most Class A models as long as the structure constants are taken in the form displayed in [15] (the exception is the group of Type VI₋₁). However, this diagonal form may not exhaust the possible physics available in these models, and we do not assume diagonality here.

Notice that the HPD cannot be used to make the initial conditions satisfy equation (6.3): Under a change

$$g_{ab} \rightarrow g'_{ab} = M_a^c g_{cd} M_b^d,$$

the momentum constraints change as

$$\mathcal{H}_a(\mathbf{g}') = (\det \mathbf{M}) M_a^b \mathcal{H}_b(\mathbf{g}).$$

The result of our analysis is the following: The completely reduced Lagrangian gives the correct equations of motion for Class A models if and only if we tune the initial conditions (at $t = 0$) $g_{ab}(0)$, $\dot{g}_{ab}(0)$ in such a way that

$$C_{ab}^c g^{bd}(0) \dot{g}_{dc}(0) = 0. \quad (6.4)$$

To this result we must add the fact that two three-metrics, g_{ab} and g'_{ab} , are physically equivalent, that is, gauge related, if there is an HPD such that in the notation of equation (5.1) $g'_{ab} = M_a^c g_{cd} M_b^d$. We must implement this fact in counting the number of independent initial conditions. In order to have a correct dynamics and a correct counting of the degrees of freedom for Class A Bianchi models we must require that (1) the initial conditions must be chosen such that equation (6.4) is satisfied, and (2) initial conditions related by HPD must be considered physically equivalent.

VII. HAMILTONIAN APPROACH

The standard Hamiltonian ADM approach [15] is built upon the Lagrangian equation (3.3). Since \mathcal{L} does not depend on \dot{N} and \dot{N}^a , the conjugate variables \mathcal{P}_0 and \mathcal{P}_a to N and N^a are the primary constraints in phase space. The canonical Hamiltonian has the form

$$H = \int d^3x (N\mathcal{H}_0 + N^a \mathcal{H}_a). \quad (7.1)$$

The Dirac Hamiltonian H_D is formed by adding to H a linear combination (with arbitrary functions λ_0, λ_a) of the primary constraints:

$$H_D = H + \int d^3x (\lambda^0 \mathcal{P}_0 + \lambda^a \mathcal{P}_a). \quad (7.2)$$

Stability of the primary constraints under the evolution generated by H_D leads to constraints which are now secondary:

$$\mathcal{H}_0 = 0, \quad \mathcal{H}_a = 0, \quad (7.3)$$

and no more constraints arise.

All eight constraints are first class, and the four independent gauge transformations that account for the diffeomorphism invariance are made out of them. For the sake of completeness let us write the gauge generators which act through the Poisson bracket in the whole phase space [19] (three-space integrations are understood for all repeated indices):

$$G(t) = \mathcal{P}_\mu \dot{\xi}^\mu + (\mathcal{H}_\mu + N^\rho D_{\mu\rho}^\nu \mathcal{P}_\nu) \xi^\mu, \quad (7.4)$$

where the functions $D_{\mu\rho}^\nu$ describe the first class structure of the secondary constraints,

$$\{\mathcal{H}_\mu, \mathcal{H}_\rho\} = D_{\mu\rho}^\nu \mathcal{H}_\nu, \quad (7.5)$$

and ξ^μ are arbitrary functions. The relationship of the transformations generated by $G(t)$ to the standard diffeomorphisms generated by a vector field $\epsilon^\mu \partial_\mu$ is given by (see [19] for instance):

$$\epsilon^0 = \frac{\xi^0}{N}, \quad \epsilon^a = \xi^a - \frac{N^a}{N} \xi^0. \quad (7.6)$$

On a given metric, every infinitesimal spacetime diffeomorphism ϵ^μ is matched by $G(t)$ with the specific functions ξ^μ dictated by equation (7.6). This means that we have exactly the same gauge group either in the Lagrangian or in the Hamiltonian formalism, here extended to the phase space, and that in this last formalism the same steps must be taken to reduce the gauge group to the time-reparameterization invariance.

The implementation of the Killing conditions and the gauge fixing $N^a = 0$ into the Hamiltonian equation (7.2) leads to the same conclusions we have arrived at in the Lagrangian formulation. In particular, only for Class A models is the implementation of the Killing conditions into the Hamiltonian correct, in the sense that it gives the correct equations of motion. The reason is that the functional derivatives in the Hamiltonian formalism develop terms like the ones in equation (3.6). In fact, any of the \mathcal{H}_μ is of the form

$$\mathcal{H}_\mu = |\omega| \tilde{\mathcal{H}}_\mu ,$$

in such a way that the three-space functional derivatives, corresponding to the Hamiltonian equations of motion, become (here X stands for $g_{ab}, \tilde{\pi}^{ab}$, where $\tilde{\pi}^{ab}$ is the canonical conjugate of g_{ab} in the reduced formalism: $\pi^{ab} = |\omega| \tilde{\pi}^{ab}$)

$$\begin{aligned} \frac{\delta \mathcal{H}_\mu}{\delta X} = & |\omega| \left(\frac{\partial \tilde{\mathcal{H}}_\mu}{\partial X} - (\mathbf{Y}_a + C_{ac}^c) \left(\frac{\partial \tilde{\mathcal{H}}_\mu}{\partial \mathbf{Y}_a X} \right) \right. \\ & \left. + \sum_{a \geq b} (\mathbf{Y}_b + C_{bd}^d) (\mathbf{Y}_a + C_{ac}^c) \left(\frac{\partial^2 \tilde{\mathcal{H}}_\mu}{\partial \mathbf{Y}_a \mathbf{Y}_b X} \right) \right) . \end{aligned} \quad (7.7)$$

(As a matter of fact, only \mathcal{H}_0 depends on second order space derivatives.)

To illustrate this last result, let us write the Poisson brackets for the momenta \mathcal{H}_a . They satisfy

$$\begin{aligned} & \left\{ \int d^3x \mathcal{H}_a \xi_1^a, \int d^3x \mathcal{H}_b \xi_2^b \right\} \\ &= \int d^3x \mathcal{H}_c (\xi_1^a \xi_{2,a}^c - \xi_2^a \xi_{1,a}^c + C_{ab}^c \xi_1^a \xi_2^b) . \end{aligned} \quad (7.8)$$

When we implement the Killing conditions on the quantity $\int d^3x \mathcal{H}_a \xi^a$, in order to get rid of the three-space dependence, it seems that we are bound to take the functions ξ^a as constants (but this is not exactly true, as we will see shortly). This would generate diffeomorphisms of the type equation (5.8), namely inner automorphisms for the left-invariant Lie algebra. Then $\tilde{\mathcal{H}}_a$ reduces to

$$\mathcal{H}_a^{\text{red}} = C_{ab}^c \tilde{\pi}^{bd} g_{dc} + C_{bc}^b \tilde{\pi}^{cd} g_{da} , \quad (7.9)$$

(this is the phase space version of equation (6.2)).

To continue, we must first introduce the Poisson brackets for the reduced theory. The reduced Poisson brackets $\{-, -\}_R$ are defined through a renormalization of the old Poisson brackets $\{-, -\}_R = V \{-, -\}$, where V is the (perhaps infinite) right invariant volume element $V = \int \omega^1 \wedge \omega^2 \wedge \omega^3 = \int d^3x |\omega|$. Notice that as long as the Killing conditions hold we can invert the relation $\pi^{ab} = |\omega| \tilde{\pi}^{ab}$ to $\tilde{\pi}^{ab} = \frac{1}{V} \int d^3x \pi^{ab}$. The commutation relations for $\mathcal{H}_a^{\text{red}}$ are

$$\begin{aligned} \{\mathcal{H}_a^{\text{red}}, \mathcal{H}_b^{\text{red}}\}_R = & C_{ab}^c \mathcal{H}_c^{\text{red}} \\ & + C_{dc}^d (C_{ab}^e \tilde{\pi}^{cf} g_{fe} + C_{fa}^f \tilde{\pi}^{ce} g_{eb} - C_{fb}^f \tilde{\pi}^{ce} g_{ea}) . \end{aligned} \quad (7.10)$$

whereas the reduction of the right side of equation (7.8) is, after factoring out the three-space volume,

$$C_{ab}^c \mathcal{H}_c^{\text{red}} .$$

It is only for Class A models that we get this result in equation (7.10): Only for Class A models do the implementation of the Killing conditions and the computation of the Poisson brackets commute.

A. Homogeneity-preserving diffeomorphisms, revisited

We have said, after equation (7.8), that the implementation of the Killing conditions on the generator $\int d^3x \mathcal{H}_a \xi^a$ does not require the functions ξ^a to be constant. The reason is as follows. In the computations in equation (7.8) we have been dropping boundary terms. This is correct if our functions ξ^a have a compact support on Σ_t . But we know from the Lagrangian analysis that some diffeomorphisms that do not vanish at the boundaries, for instance the HPD, play an important role when we perform the dimensional reduction through the Killing conditions.

Consider the generator of three-space diffeomorphisms for the reduced set of variables g_{ab}, π^{ab} ,

$$\int d^3x \mathcal{H}_a \xi^a , \quad (7.11)$$

with

$$\mathcal{H}_a = -|\omega| \sqrt{g} (|\omega|^{-1} \sqrt{g} \pi^{bc})|_b g_{ca} . \quad (7.12)$$

In order to get the standard result

$$\delta g_{ab} = \left\{ g_{ab}, \int d^3x \mathcal{H}_a \xi^a \right\} = \xi_{b|a} + \xi_{a|b} , \quad (7.13)$$

we always perform integration by parts in equation (7.11). Since the generator equation (7.11) acts locally, what we actually do in practice is to compute $\delta g_{ab}(t, x)$ using functions that are identical to ξ_a in a neighborhood of a given point but that vanish at spatial infinity. If we perform directly the integration by parts in equation (7.11) we get

$$G_3 = \int d^3x \pi^{bc} \xi_{c|b} , \quad (7.14)$$

which differs from equation (7.11) at most by a boundary term.

Boundary terms may or may not exist depending on the topology of Σ . As long as gauge transformations on local functions are considered, equation (7.14) is as good as equation (7.11) to generate the transformations equation (7.13), but G_3 is better when we must work with arbitrary functions ξ_a that are different from zero at the

boundaries. Notice that whereas equation (7.11) vanishes on a solution of the equations of motion, because \mathcal{H}_a is a constraint, equation (7.14) does not need to do so in the case when the functions ξ_a do not vanish at the boundaries. Thus G_3 is not bound to be zero on a solution of the equations of motion. It is not a constraint, except when there are no boundaries in Σ or for some other some particular cases that will be examined below.

The dimensional reductions of equation (7.11) and equation (7.14) give different result, because integration by parts for the space coordinates does not exist in the reduced formalism. Now we apply the Killing conditions to equation (7.14):

$$G_3 = \int d^3x \pi^{bc} \xi_{c|b} = \int d^3x |\omega| \tilde{\pi}^{bc} (\mathbf{Y}_b \xi_c - \Gamma_{bc}^d \xi_d),$$

where the connection coefficients Γ_{bc}^d are

$$\Gamma_{bc}^d = \frac{1}{2} (g_{ce} C_{bf}^e g^{fd} + g_{be} C_{cf}^e g^{fd} - C_{bc}^d),$$

(here $g^{fd} = e^{fd}$ since $N^a = 0$). Therefore,

$$G_3 = \int d^3x |\omega| \tilde{\pi}^{bc} g_{cd} (\mathbf{Y}_b \xi^d - C_{be}^d \xi^e). \quad (7.15)$$

Dimensional reduction requires us to factor out the volume $V = \int d^3x |\omega|$. This means that we need the remaining piece in the integrand, $\tilde{\pi}^{bc} g_{cd} (\mathbf{Y}_b \xi^d - C_{be}^d \xi^e)$, to be independent of the space coordinates. Since $\tilde{\pi}^{bc}$ and g_{cd} are already space independent because they satisfy the Killing conditions, we end up with the requirement

$$\mathbf{Y}_b \xi^d - C_{be}^d \xi^e = B_b^d, \quad (7.16)$$

with B_b^d a constant matrix. This condition is exactly equation (5.2), the condition for HPD, expressed in the dual basis. Thus we see that the dimensional reduction in phase space has room for and only for the HPD we already found in the Lagrangian formalism. Then G_3 becomes $G_3 = V B_b^d \tilde{\pi}^{bc} g_{cd}$, and since the reduced Poisson brackets are $\{-, -\}_R = V \{-, -\}$, we get for the canonical generator of HPD in the reduced formalism,

$$\tilde{G}_3 = B_b^d \tilde{\pi}^{bc} g_{cd}. \quad (7.17)$$

We find here another special feature of the Bianchi models, for the \tilde{G}_3 in equation (7.17) generates gauge transformations, HPD, and yet it is not a constraint of the reduced formalism by itself, except for the case in Class A models of inner automorphisms. In this last case, we have $B_b^d = B^c C_{cb}^d$, and \tilde{G}_3 for Class A models becomes $\tilde{G}_3 = B^a \mathcal{H}_a^{\text{red}}$, with $\mathcal{H}_a^{\text{red}}$ given in equation (7.9) with $C_{bc}^b = 0$. These results completely match the ones previously obtained in the Lagrangian analysis.

The generators of HPD that define automorphisms of the Lie algebra that are not inner automorphisms (called “outer” HPD in [9]) are always constants of motion [9] of

the reduced formalism. They are enforced to be constraints if Σ has no boundaries, although these constraints cannot be retrieved as such constraints from the reduced formalism by itself. In any case, either being constants of motion or constraints, they always generate gauge transformations in the reduced formalism. These transformations, in either the Lagrangian or the Hamiltonian picture, are available as gauge transformations from the outset, as a consequence of the rationale of the reduction procedure, but they cannot be retrieved as gauge transformations from the reduced formalism by itself.

A comment is in order for Class B models. The generators equation (7.17) give the HPD for Class B as well as Class A models. In particular, $C_{ab}^d \tilde{\pi}^{bc} g_{cd}$ generates the inner automorphisms associated, in the configuration space picture, with the right-invariant vector \mathbf{Y}_a . Nevertheless, $C_{ab}^d \tilde{\pi}^{bc} g_{cd}$ is not necessarily a constraint for the Class B models because the true momentum constraints are those of equation (7.9), as one can directly verify from the equations of motion. Oddly enough, in Class B models at least one of the generators of the residual gauge transformations associated with inner automorphisms is not a constraint. Also, according to equation (7.10), these constraints, two of them, are no longer first class. These results hint that the Lagrangian or Hamiltonian formulation for Class B models is not possible in general.

Notice that in the case of three-manifolds Σ with no boundaries, the possible new constraints \tilde{G}_3 in equation (7.17) appear also in the Lagrangian formalism by simply writing down the process from equation (7.11) to equation (7.14) in configuration-velocity variables.

One could think that the mechanism of integration by parts, as used to transform equation (7.11) into equation (7.14), and that helped us find the whole set of HPD in phase space, could perhaps be used in Class B models to transform the reduced \mathcal{H}_0 into the correct generator of time translations. In fact, some unwanted pieces can be eliminated this way, but this does not solve the problem.

B. Degrees of freedom

Notice also that the tangency of the Hamiltonian evolution operator equation (7.1) to the gauge fixing surface defined by $N^a = 0$ implies $\lambda^a = 0$. On the other hand, once the gauge fixing $N^a = 0$ has been introduced into H_D , the stability of the primary constraints $\mathcal{P}_a = 0$ will no longer produce the secondary constraints $\mathcal{H}_a = 0$. This result parallels that of equation (6.1). As a matter of fact, and since they have become second class constraints, these three couples of canonical variables N^a, \mathcal{P}_b , are readily eliminated from the formalism by taking the Dirac bracket, which is nothing but the ordinary Poisson bracket for the rest of the variables (the same number of variables, N^a, \dot{N}^b are eliminated at the same stage in the Lagrangian formulation). The important point is that the momentum constraints are gone

from the reduced formalism, but yet they must be implemented (in case they do not vanish identically) from the outset as restrictions imposed on the initial conditions, if we want to have the right equations of motion.

It is remarkable also that the generators of HPD in the dimensionally reduced theory are not necessarily constraints, except for the case of inner automorphisms and Class A models. We still need to fix all the residual gauge freedom corresponding to the three-diffeomorphisms, because these three-diffeomorphisms have been with us since the beginning. We are in the same situation as we were in the Lagrangian formulation, and therefore the same HPD, exactly the same, appear here, as we have just shown. *That is why the degrees of freedom in both the Lagrangian and the Hamiltonian formalisms coincide.*

In [9] a discrepancy between Lagrangian and Hamiltonian degrees of freedom is argued on the basis that the only HPD available in Hamiltonian formalism are those yielding inner automorphisms. Our analysis differs from the one in [9] in that we show that the residual three-diffeomorphism invariance that still needs to be fixed corresponds to the HPD that we found earlier in the Lagrangian formalism. And this is something that one knows in advance through the process of the reducing the original gauge group.

The claims in [9] are opposed in [20], where it is argued that the degrees of freedom for both the Lagrangian and Hamiltonian formalisms are the same. However, they base their claim on rejecting the HPD that are not associated with inner automorphisms as residual gauge transformations that need to be fixed. In the next section we show that all the HPD are indeed residual gauge transformations that describe redundancy in the initial conditions of the system and therefore do need to be fixed.

VIII. CONCLUSIONS

We have proved that Class A Bianchi models allow for a Lagrangian formulation as a mechanical particle-like system (finite number of degrees of freedom, in contrast to field theory) as long as the setting of the initial conditions is taken in a certain way.

In our transit toward this result we have proceeded as follows: We have carefully produced a partial gauge fixing in four steps, to reduce the initial gauge group generated by the four-diffeomorphism invariance to time reparameterization invariance. As a consequence, the troubles that beset Class B Bianchi models are identified as obstructions to the commutativity between two processes: implementing the Killing conditions either into the equations of motion or directly into the Lagrangian, followed by deriving the equations of motion.

We distinguish, as regards the Lagrangian or Hamiltonian formulations, two types of problems, which correspond, respectively, to the implementation of the Killing conditions into the Lagrangian and to the implementa-

tion of the gauge fixing that sets the shift variables to zero. As we have just said, the first problem prevents the Class B Bianchi models from having a reduced Lagrangian or Hamiltonian formulation in general. The second tells us that the initial conditions must be chosen to satisfy some former constraints (the momentum constraints) of the original theory, even though they are no longer constraints for the reduced one.

We have shown that the Hamiltonian formalism has the same gauge freedom that is available in the Lagrangian formulation. The reduction of the gauge group in phase space follows exactly the same steps as in configuration-velocity space, and the same features, the same problems, the same considerations, and the same results apply as well. The fact that the reduced momenta \mathcal{H}_a in Class A models can only generate HPD that are inner automorphisms of the Lie algebra has its counterpart in configuration-velocity space in that these HPD are the only ones that can be derived as gauge transformations from the reduced Lagrangian on its own. But that does not mean that the HPD associated with the “outer” automorphisms, if there are any, do not need to be quotiented out. Rather, all HPD must be quotiented out, either in the Hamiltonian or in the Lagrangian formalism.

Also, what is and what is not a gauge symmetry and the role of the residual gauge symmetries has been analyzed in detail. We think that our considerations throw a definitive light on some issues that have not yet been settled in the literature [9,20].

Through our analysis, the gauge group is reduced in several stages, and residual gauge symmetries appear. According to the fact that they all proceed from the original gauge group, we claim that these residual gauge symmetries must be fixed thoroughly. We think that this is an important point that deserves further elaboration below.

A. Gauge fixing for the residual gauge group

Let us show the need to fix the gauge for the residual gauge group of three-space diffeomorphisms that appears at the end of the third step of the gauge fixing procedure (Section IV). Consider that we have a solution $g_{ij}(x, t), N(t)$ of the Einstein equations with $N^i = 0$ and with initial conditions at $t = 0$. Now consider the action of the infinitesimal diffeomorphism transformation defined by the vector field $\epsilon^i \partial_i$,

$$\epsilon^i(x, t) = f(t)\tilde{\epsilon}^i(x),$$

where f has been chosen to satisfy

$$f(0) = 0, \quad \dot{f}(0) = 0; \quad f(t) = 1, \quad \dot{f}(t) = 0 \text{ for } t \geq 1.$$

Since this vector field generates a foliation-preserving diffeomorphism, it will define a gauge transformation

which is still allowed in the formalism after the first-step gauge-fixing has been done. The transformed metric $g'_{ij}(x)$, $N'(x)$, $N'^i(x)$ is also a solution of the Einstein equations. Notice that both metrics $g_{\mu\nu}$ and $g'_{\mu\nu}$ share the same initial conditions at $t = 0$. It is then obvious that the correct interpretation of a gauge transformation dictates that $g_{\mu\nu}$ and $g'_{\mu\nu}$ are physically equivalent and must be so for any other time we take for the setting of the initial conditions.

Now consider the relation between these two metrics but with initial conditions taken at $t = 1$. They are related by an infinitesimal three-space diffeomorphism generated by the time-independent vector field $\bar{\epsilon}^i(x)\partial_i$. The two metrics satisfy the gauge $N^i = 0$. The gauge transformation that connects them is not localizable in time (being different from the identity only within a finite time interval), which is the point made in [20] to deem this transformation as non-gauge, and yet both metrics must be physically identified. This result proves that we must fix the gauge even for the three-space diffeomorphisms, namely, the residual gauge group. It is in this place that the HPD play the significant role we have seen in Section V and Section VII.

Our analysis agrees with that of [9] with regard to the Lagrangian formulation, and we agree with their space-time counting of the degrees of freedom (see the table in [9]; also see [16–18] for a counting of degrees of freedom in models with compactified spatial sections). But we differ in other respects. Let us make the differences clear; they concern the status of the HPD associated with “outer” automorphisms of the right invariant Lie algebra.

In [9], “outer” HPD, that is, HPD yielding automorphisms of the Lie algebra that are not inner automorphisms, count as gauge degrees of freedom in the Lagrangian formulation, but are considered non-gauge symmetries in phase space. In [20], these “outer” HPD are always taken in both formulations as non-gauge symmetries. Instead, from our systematic procedure of reducing the gauge group, we deduce that the “outer” HPD are gauge transformations in both the Lagrangian and the Hamiltonian formalisms, and they always count as gauge degrees of freedom.

Also see our comments in Appendix D. For example, we explicitly consider the case of the Bianchi Type I model, where the number of degrees of freedom is 1 (if the surfaces of homogeneity have the topology of \mathbb{R}^3). It may appear to some people that an odd number of degrees of freedom in a Hamiltonian formalism is somehow not correct, but that is the case here, and we discuss this matter in a bit more detail in Appendix D.

Summing up, we have exhibited in a simple way the two different problems that appear when we implement the Killing conditions and the gauge fixing into the Lagrangian or into the Hamiltonian. The first problem prevents the Class B Bianchi models from having Lagrangian or Hamiltonian formulations (except possibly in special cases), whereas the second is solved by intro-

ducing some requirements on the initial conditions. We have also shown that there is no ambiguity or matter of interpretation in what must be understood as a gauge transformation, either in the Lagrangian or the Hamiltonian formalism. Finally our analysis proves that the number of degrees of freedom in both formalisms is always the same.

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APPENDIX A: GENERAL SPATIALLY HOMOGENEOUS METRIC

In order to demonstrate some of our ideas explicitly, we here give formulas for the connection, curvature, and field equations for a general spatially homogeneous metric, including a general lapse function and shift vector. A word on calculational procedure: It is easiest to use an orthonormal basis adapted to the Killing structure:

$$\mathbf{g} = \eta_{\mu\nu}\sigma^\mu\sigma^\nu = -(\sigma^0)^2 + \delta_{ij}\sigma^i\sigma^j, \quad (\text{A1})$$

where the basis is defined by

$$\sigma^0 = N\mathbf{d}t, \quad \sigma^i = b_a^i(\omega^a + N^a\mathbf{d}t), \quad (\text{A2})$$

where N, N^a, g_{ab} are functions only of t , and we use the convention that bold face denotes a tensor or a form; the one-forms ω^a obey the following relation, indicating that they are invariant under the isometry group whose Lie algebra is defined by the structure constants C_{bc}^a :

$$\mathbf{d}\omega^a = \frac{1}{2}C_{bc}^a\omega^b\wedge\omega^c. \quad (\text{A3})$$

Note that the three-metric is defined by

$$g_{ab} = \delta_{ij}b_a^ib_b^j, \quad (\text{A4})$$

and we will consistently use i, j, \dots for orthonormal indices, and a, b, c, \dots for ordinary three-indices. The matrix $\mathbf{B} = (b_a^i(t))$ represents an arbitrary square root of the metric; precisely what form it takes will be irrelevant. The inverse of the three-metric is defined by

$$e^{ac}g_{cb} = \delta_b^a \quad (\text{A5})$$

in order not to confuse it with the ab components of the contravariant four-metric. The inverse of \mathbf{B} is the matrix $\mathbf{A} = (a_i^a)$, so that

$$e^{ab} = a_i^a a_j^b \delta^{ij}, \quad \text{where } a_j^a b_a^i = \delta_j^i \iff a_i^a b_b^i = \delta_b^a. \quad (\text{A6})$$

The first Cartan equations with the torsion set equal to zero are

$$d\sigma^\mu = -\sigma^\sigma \wedge \sigma^\mu_\sigma, \quad (\text{A7})$$

where σ^μ_ν are the connection one-forms, used in forming the covariant derivative of a tensor. The metric compatibility equations (covariant derivative of the metric equals zero), when the metric components are constants, as in an orthonormal basis, here read

$$\eta_{\mu\sigma}\sigma^\sigma_\nu = -\eta_{\nu\sigma}\sigma^\sigma_\mu. \quad (\text{A8})$$

Equations (A7) and (A8) uniquely determine the connection one-forms. The second Cartan equations are

$$\frac{1}{2}R^\mu_{\nu\sigma\tau}\sigma^\sigma \wedge \sigma^\tau = d\sigma^\mu_\nu + \sigma^\mu_\sigma \wedge \sigma^\sigma_\nu; \quad (\text{A9})$$

they determine the Riemann tensor components (see [15] for more details).

It is convenient to define a matrix related to the logarithmic derivative of \mathbf{B} and its symmetric and antisymmetric parts as the matrices \mathbf{K} , \mathbf{L} , \mathbf{M} . It is also convenient to define the orthonormal projection of the structure constants (which then become a time-dependent array \mathbf{D} plus another array \mathbf{E} by (here $\dot{}$ denotes d/dt):

$$K_{ij} = \frac{1}{N}(\dot{b}_a^i a_j^a - b_a^i N^b a_j^c C_{bc}^a), \quad (\text{A10a})$$

$$L_{ij} = \frac{1}{2}(K_{ij} + K_{ji}), \quad M_{ij} = \frac{1}{2}(K_{ij} - K_{ji}), \quad (\text{A10b})$$

$$D_{jk}^i = b_a^i a_j^b a_k^c C_{bc}^a, \quad E_{ijk} = D_{jk}^i - D_{ik}^j - D_{ij}^k. \quad (\text{A11})$$

With these definitions, we have for the curls of the orthonormal basis forms

$$d\sigma^0 = 0, \quad d\sigma^i = K_{ij}\sigma^0 \wedge \sigma^j + \frac{1}{2}D_{jk}^i\sigma^j \wedge \sigma^k. \quad (\text{A12})$$

The connections forms are:

$$\sigma_i^0 = \sigma_0^i = L_{ij}\sigma^j, \quad \sigma_j^i = -M_{ij}\sigma^0 + \frac{1}{2}E_{ijk}\sigma^k. \quad (\text{A13})$$

The results for the independent components of the Riemann tensor are most conveniently displayed after raising an index (using δ^{ij}):

$$R_{0j}^{0i} = \frac{1}{N}\dot{L}_{ij} + L_{ik}L_{jk} - L_{ik}M_{jk} - L_{jk}M_{ik}, \quad (\text{A14a})$$

$$R_{k\ell}^{0i} = L_{ij}D_{k\ell}^j + \frac{1}{2}(L_{jk}E_{j\ell}^i - L_{j\ell}E_{jk}^i), \quad (\text{A14b})$$

$$R_{k\ell}^{ij} = L_{ik}L_{j\ell} - L_{i\ell}L_{jk} - \frac{1}{4}E_{imk}E_{j\ell}^m + \frac{1}{4}E_{im\ell}E_{jmk} + \frac{1}{2}E_{ijm}D_{k\ell}^m. \quad (\text{A14c})$$

The Ricci tensor components, the scalar curvature, and the Einstein tensor components are defined by

$$R_\nu^\mu = R_{\nu\sigma}^{\mu\sigma}, \quad R = R_\sigma^\sigma, \quad G_\nu^\mu = R_\nu^\mu - \frac{1}{2}R\delta_\nu^\mu.$$

The evolution equations (in vacuum) are setting the space-space components of the Ricci tensor to zero, where

$$\begin{aligned} R_j^i &= \frac{1}{N}\dot{L}_{ij} - L_{ik}M_{jk} - L_{jk}M_{ik} + L_{kk}L_{ij} \\ &+ \frac{1}{2}(D_{jk}^i + D_{ik}^j)D_{k\ell}^\ell \\ &- \frac{1}{2}D_{ik}^\ell(D_{j\ell}^k + D_{j\ell}^\ell) + \frac{1}{4}D_{k\ell}^iD_{k\ell}^j. \end{aligned} \quad (\text{A15})$$

Next we display the time-time and time-space components of the Einstein tensor; these set to zero are the constraint equations on initial value data:

$$\begin{aligned} G_0^0 &= \frac{1}{2}L_{st}L_{st} - \frac{1}{2}(L_{ss})^2 \\ &+ \frac{1}{2}D_{st}^tD_{su}^u + \frac{1}{4}D_{su}^tD_{st}^u + \frac{1}{8}D_{tu}^sD_{tu}^s, \end{aligned} \quad (\text{A16a})$$

$$G_i^0 = -L_{ij}D_{jk}^k - L_{jk}D_{ki}^j. \quad (\text{A16b})$$

Finally, we will need the scalar curvature in order to display the reduced Lagrangian. In the action integral the volume element is really

$$\sigma^0 \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 = N\sqrt{g} dt \wedge \omega^1 \wedge \omega^2 \wedge \omega^3, \quad (\text{A17})$$

where $g = \det(g_{ab})$, and the spatial integral can be set to the constant V :

$$\int \int \int \omega^1 \wedge \omega^2 \wedge \omega^3 = V. \quad (\text{A18})$$

The action integral is therefore

$$\mathcal{I} = \int \mathcal{L} V dt,$$

where

$$\begin{aligned} \mathcal{L} &= RN\sqrt{g} \\ &= (2\sqrt{g}L_{ss})' - 2\sqrt{g}N^a b_a^s D_{su}^u L_{ss} \\ &+ N\sqrt{g} \left(L_{st}L_{st} - (L_{ss})^2 \right. \\ &\quad \left. - D_{st}^tD_{su}^u - \frac{1}{2}D_{su}^tD_{st}^u - \frac{1}{4}D_{tu}^sD_{tu}^s \right), \end{aligned} \quad (\text{A19})$$

and note that we have separated out a total time derivative. Here it is best to be explicit, and we display the Lagrangian in terms of the configuration space variables N, N^a, g_{ab} :

$$\begin{aligned} \mathcal{L} = \frac{\sqrt{g}}{N} & \left[-\frac{1}{4}(e^{ab}\dot{g}_{ab})^2 + \frac{1}{4}\dot{g}_{ab}\dot{g}_{cd}e^{ac}e^{bd} \right. \\ & - \dot{g}_{ab}e^{bc}N^d C_{dc}^a + (N^a C_{ab}^b)^2 \\ & + \frac{1}{2}g_{ab}e^{cd}N^e N^f C_{ec}^a C_{fd}^b + \frac{1}{2}N^e N^f C_{eb}^a C_{fa}^b \left. \right] \\ & - N\sqrt{g} \left[+ e^{bc}C_{ba}^a C_{cd}^d + \frac{1}{2}e^{cd}C_{cb}^a C_{da}^b \right. \\ & \left. + \frac{1}{4}g_{ad}e^{be}e^{cf}C_{bc}^a C_{ef}^d \right]. \end{aligned} \quad (\text{A20})$$

The vacuum field equations are most conveniently written as four constraint equations and six evolution equations:

$$G_0^0 = 0, G_i^0 = 0, R_j^i = 0. \quad (\text{A21})$$

These equations can be derived from the Einstein-Hilbert Lagrangian density before imposing any symmetry requirements and then imposing the symmetry requirements after the equations have been derived. We will compare these equations with those derived from the reduced Lagrangian \mathcal{L} ; in other words, by imposing the symmetry requirements first and then deriving equations.

The equation $\delta\mathcal{L}/\delta N = 0$ is readily seen to be exactly $G_0^0 = 0$ (up to a factor of $2\sqrt{g}$).

The equations $\delta\mathcal{L}/\delta N^a = 0$ (after canceling N and \sqrt{g}) are

$$\begin{aligned} 0 = & +\dot{g}_{db}g^{bc}C_{ca}^d + 2C_{ac}^c N^b C_{bd}^d \\ & + g_{ef}g^{cd}N^b C_{bdl}^f C_{ac}^e + N^b C_{bc}^d C_{ad}^c. \end{aligned} \quad (\text{A22})$$

This is not proportional to the G_i^0 equation in a Class B model; the difference (up to nonzero factors) is

$$[\dot{g}_{ab}g^{bf} + 2C_{ac}^c N^f - g_{ae}g^{cf}N^b C_{bc}^e] C_{fd}^d. \quad (\text{A23})$$

In other words, if the lapse is kept in, then all of the constraint equations can be derived for the Class A models. In Class B, if these terms happen to be zero anyway, well and good, but they won't automatically vanish. Note several things about the terms in equation (A23): The first one is independent of N^a ; therefore, one might hope that adding an appropriate term to the Lagrangian might remove it (a term, that is, linear in N^a). The second term has N^a multiplied by a matrix which is symmetric; supposedly it too could be removed with an appropriate term in the Lagrangian. The last term, however, has N^a times a matrix which is not symmetric; it would seem that there is no hope of generating this term by adding something to the Lagrangian.

APPENDIX B: SPATIALLY HOMOGENEOUS MAXWELL THEORY

A relatively simple system which illustrates many of our ideas is that of a spatially homogeneous electromagnetic potential which obeys the Maxwell equations. (Here

we start with the four-vector potential rather than with a spatially homogeneous field tensor, which of course could be produced using a non-homogeneous potential vector.) The background metric is taken to be a simple spatially homogeneous one, so we do not require that it obey any particular field equations. We take the metric to have components $\eta_{\mu\nu}$ in an invariant basis:

$$\mathbf{g} = -d\mathbf{t}^2 + \delta_{ij}\boldsymbol{\omega}^i\boldsymbol{\omega}^j, \quad (\text{B1})$$

where

$$d\boldsymbol{\omega}^i = \frac{1}{2}C_{jk}^i\boldsymbol{\omega}^j \wedge \boldsymbol{\omega}^k. \quad (\text{B2})$$

Note that there is a freedom to transform the spatial basis $\{\boldsymbol{\omega}^i\}$ by an orthogonal transformation but not necessarily by a general linear transformation.

The vector potential actually is an equivalence class of one-forms related by gauge transformations. Here we demand that the class contain at least one member which is spatially homogeneous. If that one is expressed in the invariant basis, its components are functions only of t :

$$\mathbf{A} = A_0(t)d\mathbf{t} + A_i(t)\boldsymbol{\omega}^i, \quad (\text{B3})$$

and it produces the following field 2-form:

$$\mathbf{F} = d\mathbf{A} = \dot{A}_i d\mathbf{t} \wedge \boldsymbol{\omega}^i + \frac{1}{2}A_i C_{jk}^i \boldsymbol{\omega}^j \wedge \boldsymbol{\omega}^k. \quad (\text{B4})$$

Note that A_0 has disappeared; it plays no part either in the Maxwell equations or in the Lagrangian. It can be made to vanish by a gauge transformation of the kind $\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + d\lambda(t)$.

We choose to concentrate on the metric as expressed in an invariant basis; this is a choice of metric gauge. Maxwell theory cannot be expressed without at least some reference to a background spacetime metric, and so this metric gauge affects the electromagnetic potential. We also choose to concentrate on the vector potential within its equivalence class which explicitly has vanishing Lie derivative with respect to the generators of the invariance group. This, too, is a choice of gauge, though as we discuss below, in some cases there is still some residual gauge freedom.

The easiest way to calculate the Maxwell equations (in vacuum) is first to write down the dual field 2-form:

$$*\mathbf{F} = \frac{1}{2}\epsilon_{ijk}A_s C_{jk}^s d\mathbf{t} \wedge \boldsymbol{\omega}^i - \frac{1}{2}\epsilon_{ijs}\dot{A}_s \boldsymbol{\omega}^i \wedge \boldsymbol{\omega}^j, \quad (\text{B5})$$

where ϵ_{ijk} is the Levi-Civita symbol, equal to ± 1 if (ijk) is an even/odd permutation of (123) and to zero otherwise. The Maxwell equations are:

$$\begin{aligned} 0 = d*\mathbf{F} & = -\frac{1}{2}\left(\epsilon_{stu}\ddot{A}_s + \frac{1}{2}\epsilon_{ijk}A_s C_{jk}^s C_{tu}^i\right)d\mathbf{t} \wedge \boldsymbol{\omega}^t \wedge \boldsymbol{\omega}^u \\ & - \frac{1}{2}\epsilon_{isk}\dot{A}_k C_{tu}^i \boldsymbol{\omega}^s \wedge \boldsymbol{\omega}^t \wedge \boldsymbol{\omega}^u, \end{aligned} \quad (\text{B6})$$

and they are equivalently written as:

$$0 = \dot{A}_j C_{js}^s, \quad (\text{B7a})$$

$$0 = \ddot{A}_i + \frac{1}{2} A_s C_{jk}^s C_{jk}^i + A_s C_{it}^s C_{tu}^u. \quad (\text{B7b})$$

One can double-check these equations by forming the connection one-forms and computing $F^{\mu\sigma}_{;\sigma} = 0$.

The action integral for a general Maxwell vector potential is (4g is the determinant of the spacetime metric in a coordinate system):

$$\mathcal{I} = \int \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \sqrt{|{}^4g|} d^4x. \quad (\text{B8})$$

In this case, $\sqrt{|{}^4g|}$ is independent of t , and so the reduced Lagrangian \mathcal{L}_R is:

$$\mathcal{L}_R = \frac{1}{4} F^{\sigma\tau} F_{\sigma\tau} = -\frac{1}{2} \dot{A}_s \dot{A}_s + \frac{1}{4} A_s A_t C_{jk}^s C_{jk}^t. \quad (\text{B9})$$

The Euler-Lagrange equations $0 = \delta\mathcal{L}_R/\delta A_i$ are:

$$0 = \ddot{A}_i + \frac{1}{2} A_s C_{jk}^i C_{jk}^s. \quad (\text{B10})$$

It is clear that in Class B models equations (B7) and (B10) do not agree. There are two differences: Equation (B7a) is a constraint equation, and it simply cannot come from a variational principle which is homogeneous quadratic in the velocities. Equation (B7b) has the term $A_s C_{it}^s C_{tu}^u$ which does not appear in equation (B10). Notice that equation (B7a) is automatically satisfied in the Class A case ($C_{js}^s = 0$), and also equations (B7b) and (B10) then do agree.

One thing which is true, however, is that the constraint equation (B7a) is compatible both with equation (B7b) and equation (B10). To see this fact, take the time derivative of equation (B7a):

$$0 = \ddot{A}_j C_{js}^s.$$

When equation (B7b) is multiplied by C_{is}^s , the result is

$$0 = C_{is}^s \ddot{A}_i + \frac{1}{2} A_t C_{jk}^t C_{is}^s + A_j C_{it}^j C_{tu}^u C_{is}^s.$$

The middle term vanishes because of the Jacobi identity, which in a three-dimensional Lie algebra is equivalent to $0 = C_{ij}^s C_{st}^t$. The last term, which would be absent anyway if equation (B10) had been used, vanishes as a consequence of the antisymmetry of C_{it}^j in its lower indices.

Thus, even if the constraint equation (B7a) were put in by hand, the problem would remain whether any variational principle could reproduce the last term in equation (B7b).

We now turn to gauge transformations. A gauge transformation here is the addition to the vector potential of

a homogeneous one-form whose curl vanishes (and therefore which can at least locally be expressed as the curl of a function). Let this one-form be

$$\boldsymbol{\kappa} = \kappa_0 \mathbf{dt} + \kappa_i \boldsymbol{\omega}^i, \quad (\text{B11})$$

where the components κ_μ are functions only of t . We require that the curl of $\boldsymbol{\kappa}$ be zero:

$$0 = \mathbf{d}\boldsymbol{\kappa} = \dot{\kappa}_i \mathbf{dt} \wedge \boldsymbol{\omega}^i + \frac{1}{2} \kappa_i C_{jk}^i \boldsymbol{\omega}^j \wedge \boldsymbol{\omega}^k. \quad (\text{B12})$$

Therefore κ_0 is arbitrary, and κ_i must be a set of constants subject to the condition

$$\kappa_i C_{jk}^i = 0. \quad (\text{B13})$$

Only Bianchi Types VIII and IX require that κ_i be zero. In fact, in a Class B model, κ_i may be taken to be proportional to C_{ij}^j .

We illustrate with the example of the general Class B model, in which only $C_{1s}^s \neq 0$ (this prescription can always be satisfied in any Class B model by using an orthogonal transformation of the invariant basis). The Jacobi identity then requires that $C_{ij}^1 = 0$. The reduced Lagrangian in this case is

$$\mathcal{L}_R = -\frac{1}{2} (\dot{A}_1^2 + \dot{A}_2^2 + \dot{A}_3^2) + \frac{1}{4} A_A A_B C_{jk}^A C_{jk}^B, \quad (\text{B14})$$

where the indices A, B range only over (2,3). The Euler-Lagrange equations are

$$0 = \ddot{A}_1, \quad 0 = \ddot{A}_A + \frac{1}{2} A_B M_{AB},$$

where

$$M_{AB} \equiv C_{jk}^A C_{jk}^B.$$

The Maxwell Equations (B7) read

$$0 = \dot{A}_1,$$

$$0 = \ddot{A}_1, \quad 0 = \ddot{A}_A + \frac{1}{2} A_B M_{AB} + A_B N_{AB},$$

where

$$N_{AB} \equiv C_{A1}^B C_{1s}^s.$$

The reduced Maxwell equations thus differ in two ways from the Euler-Lagrange equations: First, the Maxwell equations have the constraint equation $\dot{A}_1 = 0$. The residual gauge freedom allowed by equation (B13) says that A_1 can be made zero by a choice of gauge. In contrast, the Euler-Lagrange equations only require that A_1 be at most a linear function of t , and the gauge freedom only will allow the value of A_1 to be set to zero at a particular time. It is not, however, difficult to add the constraint $\dot{A}_1 = 0$ to the Euler-Lagrange equations in an *ad hoc* manner.

Second, the Maxwell equations for A_A have the additional term involving N_{AB} . If it happens that N_{AB} is symmetric, then a Lagrangian can be found to reproduce this term: The Lagrangian would add the term $\frac{1}{2}A_A A_B N_{AB}$ to the reduced Lagrangian equation (B14). However, if N_{AB} is not symmetric, then such a Lagrangian would not in general be possible — certainly no Lagrangian could be found to reproduce the Maxwell equations exactly, though under some circumstances it may be possible to find a Lagrangian which would produce equations equivalent to the Maxwell equations.

For example, the standard structure constants of a Bianchi Type III model [1] are $C_{12}^2 = -C_{21}^2 = 1$, the other structure constants vanishing. In this case $N_{AB} = \text{diag}(1, 0)$, and the Maxwell equations can indeed be derived from a variational principle (with the constraint $0 = \dot{A}_1$ being put in by hand): The Maxwell equations are:

$$0 = \dot{A}_1 = \ddot{A}_1 = \ddot{A}_2 = \ddot{A}_3 ,$$

which clearly can come from a constrained variational principle. Note that in this Type III case the reduced Lagrangian yields as Euler-Lagrange equations:

$$0 = \ddot{A}_1 = \ddot{A}_2 + A_2 = \ddot{A}_3 ,$$

which are not at all the same as the Maxwell equations.

A second example is that of a group which has structure constants $C_{12}^2 = -C_{21}^2 = C_{13}^2 = -C_{31}^2 = 1$, the rest being zero. In fact, this group is also Type III, in a basis which is a linear transformation of the basis in the preceding example. Since this transformation is not an orthogonal one, the Maxwell equations in this case differ significantly from the preceding. The reduced Lagrangian is

$$\mathcal{L}_R = -\frac{1}{2}(\dot{A}_1^2 + \dot{A}_2^2 + \dot{A}_3^2) + A_2^2 .$$

The Euler-Lagrange equations are:

$$0 = \ddot{A}_1 = \ddot{A}_2 + 2A_2 = \ddot{A}_3 .$$

The Maxwell equations are:

$$0 = \dot{A}_1 = \ddot{A}_1 = \ddot{A}_2 = \ddot{A}_3 - A_2 .$$

The difference between the Maxwell equations and the Euler-Lagrange equations in this example are profound: First, the Maxwell equations include the constraint $\dot{A}_1 = 0$, as in the previous example. Second, the Maxwell equations cannot be derived from a variational principle, unlike the previous example.

APPENDIX C: RELATIONSHIP BETWEEN SETTING THE GAUGE AND VARIATIONAL PRINCIPLES

Here we summarize and expand the main result of [14]. For simplicity we will use the language of mechanics (finite number of degrees of freedom), although everything

can be translated to field theory. Consider a singular Lagrangian L that leads to some primary constraints, $\phi_\rho = 0$ in phase space, presumed to be effective (each has non-vanishing gradient on the constraint surface). For the sake of simplicity we will consider the case when all the constraints, primary, secondary, etc., are first class (this is what happens in our generally covariant theories). The equations of motion obtained from L are:

$$\frac{\delta L}{\delta q^i} = \alpha_i - W_{ij}\ddot{q}^j = 0 ,$$

where

$$W_{ij}\ddot{q}^j = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} , \text{ and } \alpha_i = \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \dot{q}^j - \frac{\partial^2 L}{\partial \dot{q}^i \partial t} .$$

The Lagrangian dynamics can equivalently be described by a vector field that exists on, and is tangent to, the constraint surface in configuration-velocity space:

$$\mathbf{X} \equiv \frac{\partial}{\partial t} + q^i \frac{\partial}{\partial q^i} + a^i(q, \dot{q}) \frac{\partial}{\partial \dot{q}^i} + \eta^\rho \mathbf{\Gamma}_\rho \equiv \mathbf{X}_0 + \eta^\rho \mathbf{\Gamma}_\rho . \quad (\text{C1})$$

The a^i are determined from the equations of motion and the stabilization algorithm; η^ρ are arbitrary functions of time (or spacetime in field theory) and any other variable; and $\mathbf{\Gamma}_\rho$ is

$$\mathbf{\Gamma}_\rho = \gamma_\rho^i \frac{\partial}{\partial \dot{q}^i} ,$$

where γ_ρ^i are the null vectors of W_{ij} . These null vectors can be given as

$$\gamma_\rho^i = \frac{\partial \phi_\rho}{\partial p_i}(q, \hat{p}) , \quad (\text{C2})$$

where $\hat{p}_i(q, \dot{q}) = \partial L / \partial \dot{q}^i$. Let us point out that $\eta^\rho \mathbf{\Gamma}_\rho$ in equation (C1) is the piece that corresponds in our case to equation (4.1).

Notice that $\alpha_i \gamma_\rho^i = 0$ is a consequence of the equations of motion $\delta L / \delta q^i = 0$. They are called the primary Lagrangian constraints.

Consider now a partial gauge-fixing of the dynamics given by a set of new constraints $\chi_{\rho'} = 0$, with $|\rho'| < |\rho|$, defined in configuration space (holonomic constraints). Let us split the set of indices ρ into two sets, ρ' and ρ'' , in such a way that $\mathbf{\Gamma}_{\rho''} \dot{\chi}_{\rho'} = 0$ and $|\mathbf{\Gamma}_{\rho'} \dot{\chi}_{\rho'}| \neq 0$. Then, the requirement $\mathbf{X}(\dot{\chi}_{\rho'}) = 0$ determines the functions $\eta^{\rho'}$ and leaves completely undetermined $\eta^{\rho''}$.

In such a situation, the following result is proved in [14]: If we plug the holonomic gauge-fixing $\chi_{\rho'} = 0$ into the original Lagrangian L to get the reduced Lagrangian

L_{GF} , then the following equivalence holds (where $[L]$ is the Euler-Lagrange variation):

$$[L] = 0, \quad \chi_{\rho'} = 0 \iff [L_{\text{GF}}] = 0, \quad \alpha_i \gamma_{\rho'}^i = 0, \quad (\text{C3})$$

where $\alpha_i \gamma_{\rho'}^i$ are a subset of the primary Lagrangian constraints for L with the understanding that the gauge fixing has been plugged into them.

So we see that to describe the same motions in the reduced space it is not enough in general to impose the new Lagrangian equations of motion, $[L_{\text{GF}}] = 0$, but some additional constraints must be required, too.

Now, we will prove, using the machinery of [14], that in order for the solutions of $[L_{\text{GF}}] = 0$ to satisfy the constraints $\alpha_i \gamma_{\rho'}^i = 0$, we only need to impose them in the initial conditions.

Since the gauge-fixing constraints $\chi_{\rho'} = 0$ are holonomic, they just reduce the configuration space. We have adopted the notation q^i for the local coordinates in the original configuration space. We will adopt the notation Q^a for the local coordinates in the reduced configuration space.

It is proved in [14] that the Lagrangian evolution operator in the reduced velocity space takes the form

$$\begin{aligned} \mathbf{X}_R &\equiv \frac{\partial}{\partial t} + \dot{Q}^a \frac{\partial}{\partial Q^a} + a^a(Q, \dot{Q}) \frac{\partial}{\partial \dot{Q}^a} + \tilde{\eta}^{\rho''} \tilde{\Gamma}_{\rho''} \\ &\equiv \mathbf{X}_{R0} + \tilde{\eta}^{\rho''} \tilde{\Gamma}_{\rho''}; \end{aligned} \quad (\text{C4})$$

where $\tilde{\eta}^{\rho''}$ are arbitrary functions; $\tilde{\Gamma}_{\rho''}$ is

$$\tilde{\Gamma}_{\rho''} = \tilde{\gamma}_{\rho''}^a \frac{\partial}{\partial \dot{Q}^a};$$

and similarly to equation (C2), $\tilde{\gamma}_{\rho''}^a$ is defined by

$$\tilde{\gamma}_{\rho''}^a = \frac{\partial \tilde{\phi}_{\rho''}}{\partial P_a}(Q, \hat{P}), \quad (\text{C5})$$

where $\tilde{\phi}_{\rho''}$ are the primary Hamiltonian constraints corresponding to the reduced theory. It turns out [14] that these constraints are related to the original constraints $\phi_{\rho''}$ by

$$\phi_{\rho''}(q(Q), p) = \tilde{\phi}_{\rho''}(Q, p \frac{\partial q}{\partial Q}). \quad (\text{C6})$$

From all these results, the following equalities hold:

$$\begin{aligned} \tilde{\Gamma}_{\rho''}((\alpha_i \gamma_{\rho'}^i)|_{q(Q)}) &= \frac{\partial \tilde{\phi}_{\rho''}^a}{\partial P_a} \frac{\partial ((\alpha_i \gamma_{\rho'}^i)|_{q(Q)})}{\partial \dot{Q}^a} \\ &= (\Gamma_{\rho''}(\alpha_i \gamma_{\rho'}^i))|_{q(Q)} \\ &= \mathcal{FL}^* \{\phi_{\rho'}, \phi_{\rho''}\}|_{q(Q)} = 0, \end{aligned} \quad (\text{C7})$$

Here \mathcal{FL}^* is the pullback of the Legendre map \mathcal{FL} from velocity space to phase space. We have used in the last

equality the fact that the pullback of a primary Hamiltonian constraint is identically zero.

With this new result, $\tilde{\Gamma}_{\rho''}((\alpha_i \gamma_{\rho'}^i)|_{q(Q)}) = 0$, we can see that the arbitrary part in the reduced Lagrangian evolution operator equation (C4) has no effect on $(\alpha_i \gamma_{\rho'}^i)|_{q(Q)}$. This means that in the constraint surface for the reduced theory (this surface is determined by the equations of motion for L_{GF}), the operator \mathbf{X}_{R0} must also be tangent to the surface $((\alpha_i \gamma_{\rho'}^i)|_{q(Q)}) = 0$. (This is the only way to ensure the equivalence in equation (C3), for if the operator \mathbf{X}_{R0} is not tangent to the surface $((\alpha_i \gamma_{\rho'}^i)|_{q(Q)}) = 0$, then there will be no solutions of the equations of motion for the original Lagrangian in the gauge $\chi_{\rho'} = 0$, and we know that these solutions exist.) Therefore, if we consider a solution of $[L_{\text{GF}}] = 0$ with initial conditions satisfying $(\alpha_i \gamma_{\rho'}^i)|_{q(Q)} = 0$, then the whole solution satisfies $(\alpha_i \gamma_{\rho'}^i)|_{q(Q)} = 0$.

APPENDIX D: DEGREES OF FREEDOM

In this Appendix we illustrate with a very simple case the counting of degrees of freedom for our models. We also discuss the fact that spacetime rigid symmetries may be understood as residual gauge transformations.

We start by counting the true degrees of freedom for the Bianchi Type I case. The counting can be done in configuration-velocity space or in phase space — the counting is the same [13]. We start with the variables N and \dot{N} , N^a and \dot{N}^a , g_{ab} and \dot{g}_{ab} : 20 apparent degrees of freedom. The shift vector variables may be eliminated by our general considerations, so that 14 apparent degrees of freedom are left.

There is one constraint in the Lagrangian formalism. To fix the time reparameterization invariance we define the time parameter as a function of our dynamical variables, excluding N . Its stability, that is, the fact that the time derivative of this definition must vanish, will give a new constraint, containing the variable N , with no time dependence. The stability of this last constraint gives a new constraint that can be used to isolate \dot{N} . The requirement of stability now determines the arbitrary function in the Lagrangian evolution operator, and no more constraints appear. We are left with 1+3 constraints (one true constraint plus three gauge fixing constraints), lowering the number of degrees of freedom to 10. The same counting can be done in phase space: In this case, there are 2 true constraints, and we must introduce 2 gauge fixing constraints.

In Type I the momentum constraints are identically zero, so they do not reduce the degrees of freedom. We are only left with HPD symmetries. Every constant matrix B_b^a of equation (7.16) defines an automorphism of the Lie algebra.

Now topology enters the picture. If the topology of Σ is “open” (for example, \mathbb{R}^3), simply connected, and with global Killing vectors, then to each matrix B_b^a there

corresponds an HPD through equation (5.2). Therefore, there are nine HPD gauge degrees of freedom. The generators of HPD, equation (7.17), are not constraints, because in this case equations (7.11) and (7.14) differ by boundary terms. So the final number of degrees of freedom for a Bianchi Type I model with surfaces of homogeneity with the topology of \mathbb{R}^3 is $10 - 9 = 1$.

It may appear strange to some people that a Hamiltonian formalism can turn out to have an odd number of degrees of freedom, as in the above example. In fact, we have already said that HPD need only be implemented in the initial conditions, so in this sense we have a Hamiltonian formulation plus an equivalence relation coming from the outset: It is not generated by the reduced formalism but is a remnant of the generally covariant theory we started with. To find the number of degrees of freedom number is a matter of counting constraints, which trajectories count as the “same” as others, and whether a parameterized trajectory or its orbit (its one-dimensional set of points in phase space) should be considered as physically significant. We have been guided by the principle that the Lagrangian and Hamiltonian formalisms should be equivalent (see [13]) in coming to the conclusion that in fact they are.

If the topology of Σ is that of a three-torus, then there are no infinitesimal HPD available (there are finite HPD but they do not reduce local degrees of freedom). We end up with a final number of 10 degrees of freedom. We will not consider other topologies.

Let us turn to the second consideration in this Appendix, the possibility of interpreting rigid spacetime symmetries as residual gauge transformations. Consider a free particle in \mathbb{R}^4 in a gravitational background. The action is

$$S = \int (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{1/2} d\tau, \quad (\text{D1})$$

What are the degrees of freedom? If we start with a generally covariant theory, we know that diffeomorphisms correspond to gauge degrees of freedom. Let us consider the metric background as non-dynamical, take the passive view for the action of diffeomorphisms, and consider the simplified case where there exists a system of spacetime coordinates such that the metric is just Minkowski. Now we can make the following gauge fixing: We decide to stick with Minkowskian coordinates and only allow further diffeomorphisms if they keep this condition. We end up with Poincaré transformations as residual gauge transformations.

To count the degrees of freedom, we start with positions x^μ and velocities \dot{x}^μ , which amount to 8 degrees of freedom. To fix the τ reparameterization invariance we must spend two constraints (for instance $x^0 - \tau = 0$ and its stabilization $\dot{x}^0 = 1$). We are left with 6 degrees of freedom. The residual gauge freedom consists of Lorentz transformations, but our gauge fixing forbids boosts and time translations, so we are left with three-translations

and rotations. Rotations only affect 2 degrees of freedom, because the norm of the velocity three vector is unchanged, and therefore we eliminate 5 of the 6 remaining degrees of freedom. We end up with a single degree of freedom that in our gauge fixing corresponds to the kinetic energy.

Here we see a matter of interpretation as to what is and what is not a true degree of freedom. From our point of view, if the Lorentz invariance may be considered as the residual invariance found after a process of reducing the gauge group of general covariance, then the Lorentz degrees of freedom must be considered as gauge. But as residual gauge symmetries, they are not associated to constraints but to ordinary constants of motion.

Summing up, the analysis of degrees of freedom, either in the case of Bianchi cosmologies or the simple case of a Minkowskian free particle, depends upon the point of view adopted. If one sticks to the formalism by itself, that is to say, to what the given Lagrangian yields as constraints, gauge transformations, and so on, one does not get the same number of degrees of freedom as if one considers that our theory comes from the reduction of an originally generally covariant theory. In this second point of view (which is the one we support) there are some symmetry transformations that are residual gauge transformations, that is, remnants of the reduction procedure of the Lagrangian and the gauge group, which require gauge fixing.

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